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ON GENERALIZED HYDRODYNAMIC EQUATIONS USED IN HEAT TRANSFER THEORY

V. A. BUBNOV

Heat and Mass Transfer Institute, Byelorussian Academy of Sciences, Minsk, B.S.S.R., U.S.S.R.

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Abstract—In the paper a new method is presented to compose generalized solutions of hydrodynamics equations. In the derivation of a viscous fluid equation, discontinuous change of the velocity is taken into account that occurs in shock waves, eddy flows, sonic and supersonic flows. This discontinuity is not included into hydrodynamic equations in the Navier–Stokes form.

The present method is illustrated by the solution of the problem on interaction of a vortex fibre with the surface. In this particular case the Euler equations result in the known paradox of Felix Klein and the Navier–Stokes equations give paradox solutions examined originally by M. A. Goldshtik.

The present method eliminates the above paradoxes.

NOMENCLATURE

u, v, w,	velocity vector component along the axis x, y, z;
x_0, y_0, z_0,	set of smoothing points;
T,	kinetic energy of elementary mesh;
ρ,	mesh density;
Re,	Reynolds number;
Π,	potential energy of elementary mesh.

1. INTRODUCTION

AMONG the workers who analysed motion of finite amplitude plane waves Riemann was the first who encountered with transformation of some derivatives of the differential equation into infinity and formation of shock waves. Riemann demonstrated that in this case the initial differential equation becomes senseless as for its derivation the desired functions and their derivatives had been assumed smooth. He showed also that in very particular cases only (these were later called the Hugoniot compatibility conditions) gas motion strictly follows the differential equation.

In their attempts to find the source of this contradiction mathematicians tried to revise the classical definition of the function derivative. Indeed, the classical theory of differential equations involves the problems dealing with smooth functions. So, to check whether the function $u(x, t)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0$$

it is necessary to evaluate the derivatives entering into the equation. Sobolev's attempt to withdraw the demand of smoothness from $u(x, t)$ resulted in the concept of a generalized solution. The function $u(x, t)$ which is not necessarily smooth is called a generalized solution if for any finite smooth function φ the equality

$$\iint \left(u \frac{\partial \varphi}{\partial t} - u \frac{\partial \varphi}{\partial x} \right) \mathrm{d}x \, \mathrm{d}t = 0 \qquad (1.1)$$

holds.

Equality (1.1) is the original algorithm for finding the solutions of $u(x, t)$ in a generalized sense.

In the derivation of the above equation the derivative of the function $u(x, t)$ is actually transferred to any smooth function φ.

If the initial equation (1.1) is nonlinear, then its derivation involves some difficulties reported by S. K. Godunov [1].

Let us consider a very simple nonlinear equation

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = 0.$$

If it is written as

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

then the generalized solution may be defined as a finite summable function $u(x, t)$ for any φ satisfying the identity

$$\iint \left(u\frac{\partial \varphi}{\partial t} - \frac{u^2}{2}\frac{\partial \varphi}{\partial x}\right) \mathrm{d}x\, \mathrm{d}t = 0. \qquad (1.2)$$

The first problem is encountered in treating the above equations is as follows. The same equation may be written as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) - \frac{\partial}{\partial x}\left(\frac{1}{3}u^3\right) = 0$$

that results in a different definition of a generalized solution

$$\iint \left(\frac{u^2}{2}\frac{\partial \varphi}{\partial t} - \frac{u^3}{3}\frac{\partial \varphi}{\partial x}\right) \mathrm{d}x\, \mathrm{d}t = 0. \qquad (1.3)$$

Many examples may be found which evidence that some solutions in the sense of equation (1.2) are not the solutions in the sense of equation (1.3) and vice versa, that is definitions (1.2) and (1.3) are not equivalent.

The second problem which is passed over by mathematicians in silence. It arises because due to arbitrariness of functions $\varphi(x, t)$, we distort the initial differential equation thus contravening the laws of nature.

In the present paper a new method to obtain generalized solutions is suggested which does not involve the discussed problems.

2. PARTICULAR OPERATIONS WITH DISCONTINUOUS FUNCTIONS

Let $f = f(x)$ have a number of alternating discontinuities of the first kind, that makes suitable its finite-difference representation. To do this, the whole section of the changing argument x is to be devided into i parts following the rule $x_i = x_0 + ih$. As to the behaviour of the function $f(x_i)$, it is assumed constant within each grid mesh and changes in a stepwise manner at the grid nodes (Fig. 1).

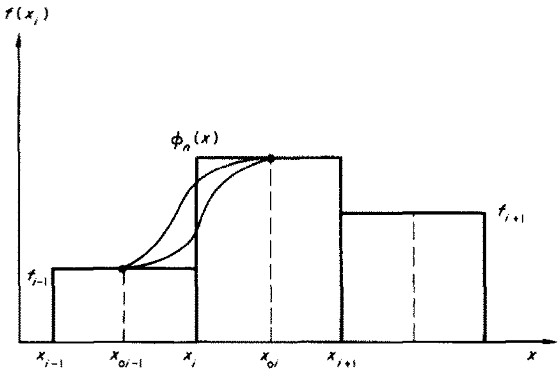

FIG. 1.

If the central coordinates of each mesh $x_{0i} = x_i + \frac{1}{2}(x_{i+1} - x_i)$ are introduced, then the discontinuous function may be approximated by a system of smooth functions $\varphi_n(x)$ within the range (x_{0i-1}, x_{0i}). It should be noted that the functions $\varphi_n(x)$ are to contact the sections of the straight line at the points x_{0i-1}, x_{0i}. The behaviour of these functions within the interval (x_{0i-1}, x_{0i}) is unimportant as finite-difference sections does not convolute into points. The functions $\varphi_n(x)$ are evident to have inflections inside the section (x_{0i-1}, x_{0i}).

N. P. Kasterin was the first who used such smoothing discontinuous function [2].

In accordance with the theory of generalized functions $\varphi_n(x)$ $(n = 1, 2, 3, \ldots)$ forms the space of main functions for the function $f = f(x_i)$.

The section (x_{i-1}, x_{i+1}) is called an interval of the function $f(x_i)$, and (x_{0i-1}, x_{0i}) is the carrier of the function $\varphi_n(x)$.

To satisfy the above conditions, it is necessary to assume

$$f(x_i) = \begin{cases} f_{i-1} = \text{const},\ x_{i-1} \leqslant x \leqslant x_i \\ f_i = \text{const},\ x_i \leqslant x \leqslant x_{i+1} \end{cases} \tag{2.1}$$

$$\varphi_n(x) = \begin{cases} f_{i-1} = \text{const},\ x \leqslant x_{0i-1} \\ f_i = \text{const},\ x \geqslant x_{0i} \end{cases}$$

$$\varphi_n(x_i) = \begin{cases} a,\ f_i > f_{i-1} \\ -a,\ f_i < f_{i-1}. \end{cases} \tag{2.2}$$

The condition of adjoining the sections of the straight-line is equivalent to zero derivatives at the points x_{0i} and x_{0i-1}, i.e.

$$\frac{d^{(n)}\varphi_n(x)}{dx^n} \equiv 0 \quad \text{if } x_{0i} \leqslant x \leqslant x_{0i-1}. \tag{2.3}$$

A linear continuous functional defined by

$$(f, \varphi) = \int_{x_{0i-1}}^{x_{0i}} f(x_i)\varphi_n(x)\,dx = f_{i-1} \times \int_{x_{0i-1}}^{x_i} \varphi_n(x)\,dx + f_i \int_{xi}^{x_{0i}} \varphi_n(x)\,dx \tag{2.4}$$

will be referred to as a generalization of the discontinuous function $f(x_i)$.

For example, if

$$\int_{x_{0i-1}}^{x_i} \varphi_n(x)\,dx = \int_{x_i}^{x_{0i}} \varphi_n(x)\,dx = \tfrac{1}{2}$$

then

$$(f, \varphi_n) = \frac{f_{i-1} + f_i}{2}. \tag{2.5}$$

To define the derivative of the generalized function $f(x_i)$, compose the functional

$$(f', \varphi_n) = \int_{x_{0i-1}}^{x_{0i}} f'(x_i)\varphi_n(x)\,dx$$

$$= f\varphi_n \Big|_{x_{0i-1}}^{x_{0i}} - \int_{x_{0i-1}}^{x_{0i}} f\varphi_n'(x)\,dx \equiv f_i^2 - f_{i-1}^2 - \left[f_{i-1} \int_{x_{0i-1}}^{x_i} \varphi_n'(x)\,dx + f_i \int_{x_i}^{x_{0i}} \varphi_n'(x)\,dx\right] =$$

$$= (f_i - f_{i-1})\,\varphi_n(x_i). \tag{2.6}$$

Here for computation of formula (2.6) the properties of formulas (2.1) and (2.2) were used.

This equality will be used as the basis for a general definition of the derivative of the generalized function.

The subsequent derivatives are easily found

$$\frac{d^2(f, \varphi_n)}{dx^2} = -(f_i - f_{i-1})\,\varphi_n'(x_i),$$

$$\frac{d^3(f, \varphi_n)}{dx^3} = (f_i - f_{i-1})\varphi_n''(x_i),$$

$$\frac{d^n(f, \varphi_n)}{dx^n} = (-1)^{n-1}(f_i - f_{i-1})\varphi_n^{(n-1)}(x_i). \tag{2.7}$$

If the absolute value of the first finite-difference derivative is introduced

$$\frac{d f_{i-1}}{d x_{0i-1}} = \frac{f_i - f_{i-1}}{(x_{0i} - x_{0i-1})} \tag{2.8}$$

then formulas (2.7) may be rewritten as

$$\frac{d^{(n)}(f, \varphi_n)}{d x^n} = \alpha_{n-1} \frac{d f_{i-1}}{d x_{0i-1}} \tag{2.9}$$

where

$$\alpha_{n-1} = (-1)^{n-1}(x_{0i} - x_{0i-1})\varphi^{(n-1)}(x_i). \tag{2.10}$$

Thus, generalized derivatives of discontinuous function $f(x_i)$ are expressed within a factor in terms of the first finite-difference derivative. It may be easily seen from formula (1.10) that the factor α_{n-1} determined the direction of the jump of a discontinuous function and may therefore be positive or negative.

3. ON HYDRODYNAMIC EQUATIONS OF DISCONTINUOUS FLOWS

Professor N. P. Kasterin who worked at the Moscow State University raised the question of revising the Euler form of hydrodynamic equations as far back as 1937. Kasterin was the first who stated and developed the idea that vortices are generated due to discontinuities in ideal fluid rather than to viscous forces. However, his mathematical manipulations were only based upon physical intuition. In what follows it will be demonstrated that Kasterin's equations for ideal fluid represent a particular case of more general equations given by the present author.

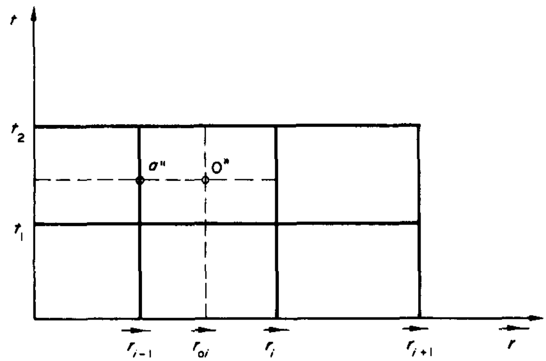

FIG. 2.

In a space filled with liquid an elementary grid mesh is cut out (Fig. 2) so that its dimension depends on the value $\boldsymbol{r}_i - \boldsymbol{r}_{i-1}$ where $\boldsymbol{r}$ is the radius vector with the coordinates x, y, z.

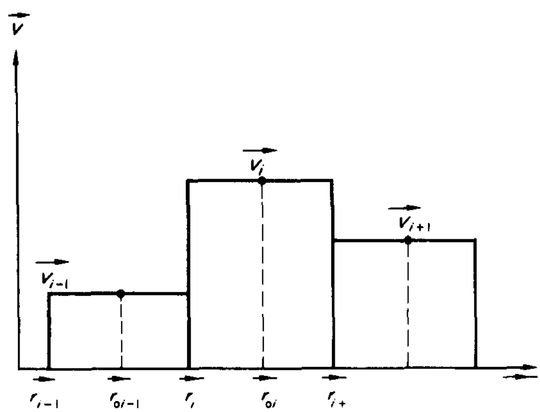

FIG. 3.

Further, the hydrodynamic velocity vector $\boldsymbol{V}(\boldsymbol{r}_i)$ is assumed to be a discontinuous function that behaves as the function $f(x_i)$ (Fig. 3).

Let us consider the i-th grid mesh $[\boldsymbol{r}_i, \boldsymbol{r}_{i+1}]$. Its velocity may be expanded into a Taylor series if formula (2.9) is used

$$\boldsymbol{V}_i = \boldsymbol{V}_{i-1} + \Delta x_{0i}\alpha_0 \frac{\partial \boldsymbol{V}_{i-1}}{\partial x_{0i-1}} + \Delta y_{0i}\alpha_0 \frac{\partial \boldsymbol{V}_{i-1}}{\partial y_{0i-1}} + \Delta z_{0i}\alpha_0 \frac{\partial \boldsymbol{V}_{i-1}}{\partial z_{0i-1}}$$

where

$$\left.\begin{aligned} \Delta x_{0i} &= x_{0i} - x_{0i-1} \\ \Delta y_{0i} &= y_{0i} - y_{0i-1} \\ \Delta z_{0i} &= z_{0i} - z_{0i-1}. \end{aligned}\right\} \tag{3.1}$$

Introduce the value α_i which is the ratio of the dimensions of two meshes

$$\alpha_i = \frac{x_{i+1} - x_i}{x_i - x_{i-1}} = \frac{y_{i+1} - y_i}{y_i - y_{i-1}} = \frac{z_{i+1} - z_i}{z_i - z_{i-1}}.$$

If a grid mesh is composed so that

$$\begin{aligned} x_{i+1} - x_{i-1} &= 2(x_{0i} - x_{0i-1}) \\ y_{i+1} - y_{i-1} &= 2(y_{0i} - y_{0i-1}) \\ z_{i+1} - z_{i-1} &= 2(z_{0i} - z_{0i-1}) \end{aligned}$$

then formulas (3.1) may be rewritten as

$$\begin{aligned} \Delta x_{0i} &= (1 + \alpha_i)(x_{0i-1} - x_{i-1}) \\ &= (1 + \alpha_i)\Delta \bar{x}_{0i} \\ \Delta y_{0i} &= (1 + \alpha_i)(y_{0i} - y_{i-1}) \\ &= (1 + \alpha_i)\Delta \bar{y}_{0i} \\ \Delta z_{0i} &= (1 + \alpha_i)(z_{0i-1} - z_{i-1}) \\ &= (1 + \alpha_i)\Delta \bar{z}_{0i}. \end{aligned} \tag{3.2}$$

Now the Taylor series for the function of $\boldsymbol{V}_i$ will be rewritten as

$$\boldsymbol{V}_i = \boldsymbol{V}_{i-1} + \beta\left(\Delta \bar{x}_{0i} \frac{\partial \boldsymbol{V}_{i-1}}{\partial x_{0i-1}} + \Delta \bar{y}_{0i} \frac{\partial \boldsymbol{V}_{i-1}}{\partial y_{0i-1}} + \Delta \bar{z}_{0i} \frac{\partial \boldsymbol{V}_{i-1}}{\partial z_{0i-1}}\right) \tag{3.3}$$

where β denotes

$$\begin{aligned} \beta &= (1 + \alpha_i)(x_{0i} - x_{0i-1})\varphi^{(0)}(x_i) \\ &= (1 + \alpha_i)(y_{0i} - y_{0i-1})\varphi^{(0)}(y_i) \\ &= (1 + \alpha_i)(z_{0i} - z_{0i} - z_{0i-1})\varphi^{(0)}(z_i). \end{aligned}$$

Change of the velocity from V_{i-1} to V_i causes additional rotation of all the points of the mesh with respect to the center of gravity of the neighbouring mesh with the angular velocity $\omega_{i-1} = \alpha_0 \operatorname{rot} V_{i-1}$. A mean linear velocity of all points of the grid mesh due to the additional rotation is then

$$V_i^1 = \alpha_0 \operatorname{rot} V_{i-1} \times (r_{0i} - r_{0i-1})$$
$$= \beta \operatorname{rot} V_{i-1} \times (r_{0i-1} - r_{i-1}). \quad (3.4)$$

Further, the volume expansion rate of the mesh under deformation $(r_{0i} - r_{0i-1})$ is

$$V_i^2 = \beta(r_{0i-1} - r_{i-1}) \operatorname{div} V_{i-1}. \quad (3.5)$$

Then the total velocity of the i-th grid mesh will be equal to

$$V = V_i + V_i^1 + V_i^2. \quad (3.6)$$

If m is used to denote the mass of the mesh, then its kinetic energy is

$$T = \frac{mv^2}{2} = m\left\{\frac{v_{i-1}^2}{2} + \beta\left(\frac{\Delta\bar{x}_{0i}}{2}\frac{\partial v_{i-1}^2}{\partial x_{0i-1}} + \frac{\Delta\bar{y}_{0i}}{2}\frac{\partial v_{i-1}^2}{\partial y_{0i-1}} + \frac{\Delta\bar{z}_{0i}}{2}\frac{\partial v_{i-1}^2}{\partial z_{0i-1}}\right) + \beta(r_{0i-1} - r_{i-1})(\operatorname{rot} V_{i-1} \times V_{i-1} + V_{i-1} \operatorname{div} V_{i-1})\right\} \quad (3.7)$$

the squared velocity vector being defined in terms of projections u, v, w as

$$V_{i-1}^2 = u_{i-1}^2 + v_{i-1}^2 + w_{i-1}^2.$$

Let the grid mesh be in a homogeneous field of surface forces. Then its potential energy is defined by the known formula

$$\Pi = -Fr_{0i}.$$

Surface forces are ordinarily prescribed by their distribution density over the surface or by the stresses

$$P = \lim \frac{\Delta F}{\Delta\sigma}; \Delta\sigma \to 0.$$

Now the following inverse formulas for the vector projection are valid

$$F_x = P_x\Delta y\Delta z = \frac{m}{\rho}\frac{\partial P_x}{\partial x};$$
$$F_y = \frac{m}{\rho}\frac{\partial P_y}{\partial_y}; F_z = \frac{m}{\rho}\frac{\partial P_z}{\partial z};$$

where

$$\rho = \frac{m}{\Delta x\Delta y\Delta z}$$

is the grid mesh density. Then the potential energy formula will be

$$\Pi = -\frac{m}{\rho}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) r_{0i}. \quad (3.8)$$

Introduction of the generalized coordinates x_{0i-1}, y_{0i-1}, z_{0i-1} and generalized velocities

$$\frac{dx_{0i-1}}{dt} = u_{i-1}; \frac{dy_{0i-1}}{dt} = v_{i-1}; \frac{dz_{0i-1}}{dt} = w_{i-1}$$

transforms the system of Helmholz equations for an elementary volume to

$$\left.\begin{aligned} \frac{\partial H}{\partial x_{0i-1}} - \frac{d}{dt}\frac{\partial H}{\partial u_{i-1}} &= 0 \\ \frac{\partial H}{\partial y_{0i-1}} - \frac{d}{dt}\frac{\partial H}{\partial v_{i-1}} &= 0 \\ \frac{\partial H}{\partial z_{0i-1}} - \frac{d}{dt}\frac{\partial H}{\partial w_{i-1}} &= 0 \end{aligned}\right\} \quad (3.9)$$

where $H = \Pi - T$ is the kinetic potential. With account for equations (3.7) and (3.8), the system (3.9) becomes

$$\frac{\partial V_{i-1}}{\partial t} + (1-\beta)(V_{i-1}\nabla)V_{i-1} - \beta V_{i-1} \operatorname{div} V_{i-1} = \frac{1}{\rho}\left(\frac{\partial P_x}{\partial x} + \frac{\partial P}{\partial y} + \frac{\partial P_z}{\partial P_z}\right).$$

As the above equality is valid for any two neighbouring meshes, the superscripts $(i-1)$ may be omitted. Then use of Stokes' hypothesis gives

$$\rho\frac{\partial V}{\partial t}+\rho[(1-\beta)(V\nabla)V-\beta V\operatorname{div}V]$$
$$=-\operatorname{grad}p+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial V}{\partial x}+\operatorname{grad}u\right)\right]$$
$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial V}{\partial y}+\operatorname{grad}v\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial V}{\partial z}\right.\right.$$
$$\left.\left.+\operatorname{grad}w\right)\right]+\operatorname{grad}(\lambda\operatorname{div}V). \tag{3.10}$$

Equation (3.10) was originally obtained on the molecular-kinetic basis by A. S. Predvoditelev [3]. If viscosity is neglected and the product $V\operatorname{div}V$ is assumed to be zero, then Kasterin's equation is obtained for the case $\beta=2$.

It should be noted that if $V\neq 0$, then assumption of a zero product is equivalent to substitution of a solid for an elementary fluid mesh at the place of discontinuity.

4. EXAMPLE OF GENERALIZED SOLUTION

Now consider the problem of an infinite plane interacting with an infinite vortex column which passes through the coordinate origin normally to the plane. If there were no plane the motion should be governed by the law

$$v=\frac{c_0}{r};\ p=p_\infty-\rho\frac{c_0^2}{2r^2}$$

where v is the tangential velocity component, p_∞ is the pressure at the infinity. The vertical w and radial u velocities would be zero in this case. Friction of the flow against air results in secondary flows which may be of interest for studying sand-storms, water spouts and hurricanes.

It is of interest to note that treatment of this problem on the basis of the Euler equations leads to Felix Klein's paradox. Klain has found that within the framework of ideal fluid, infinitely large energy is required to sustain a vortex column. It might seem that viscosity should eliminate the said paradox. However, treatment of this problem in terms of the Navier–Stokes equation made by M. A. Goldshtik [4] has shown that for Reynolds numbers above 8 the discussed problem has no finite solution.

Assuming incompressible steady axisymmetric fluod flow, Predvoditelev's equations (3.10) may be written in terms of cylindrical coordinates

$$(1-\beta)\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$+\nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$(1-\beta)\left(u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uw}{r}\right)$$
$$=\nu\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial(rv}{\partial r}+\frac{\partial^2 v}{\partial z^2}\right)$$
$$(1-\beta)\left(u\frac{\partial w}{\partial z}+w\frac{\partial w}{\partial z}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial z}$$
$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\right] \tag{4.1}$$
$$\frac{\partial(ru\partial}{\partial r}+\frac{\partial(rw)}{\partial z}=0.$$

The boundary conditions are as follows

$$u=v=w=0 \text{ at } z=0$$
$$v=\frac{c_0}{r};\ p=p_\infty-\frac{\rho c_0^2}{2r^2}\text{ at } z=\infty.$$

At the infinity the fluid is stagnant

$$u=v=w=0;\ p=p_\infty \text{ at } r=\infty.$$

At $r=0$ the component w is finite, and $u=0$ (no sources or sinks on the axis).

Introduce the dimensionless functions

$$\bar{u}=\frac{ru}{c_0};\ \Phi=\frac{rv}{c_0};\ \bar{w}=\frac{rw}{c_0};\ \pi=\frac{r^2(p-p_\infty)}{\rho c_0^2} \tag{4.2}$$

(the bar above the quantities will further be omitted).

System (4.1) is then transformed to

$$(1-\beta)\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{u^2+\Phi^2}{r}\right)= -\frac{\partial \pi}{\partial r}+\frac{2\pi}{r}+\frac{v}{c_0}r\left(\frac{\partial^2 u}{\partial r^2}-\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right);$$

$$(1-\beta)\left(u\frac{\partial \Phi}{\partial r}+w\frac{\partial \Phi}{\partial z}\right)=\frac{v}{c_0}r\left(\frac{\partial^2 \Phi}{\partial r^2}-\frac{1}{r}\frac{\partial \Phi}{\partial z^2}+\frac{\partial^2 \Phi}{\partial z^2}\right);$$

$$(1-\beta)\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}-\frac{uw}{r}\right)=-\frac{\partial \pi}{\partial z}+\frac{v}{c_0}r\left(\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w}{\partial r}+\frac{w}{r^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}=0. \tag{4.3}$$

The boundary conditions will respectively become

$$\begin{aligned} u=\Phi=w=0 \quad & \text{at } z=0\\ \Phi=1;\pi=-\tfrac{1}{2} \quad & \text{at } z=\infty\\ u=w=0 \quad & \text{at } r=0. \end{aligned}$$

The problem under discussion belongs to the class of similar problems. We therefore pass to the variable $\eta = z/r$, bearing in mind that

$$\frac{\partial r}{\partial r}=-\frac{z}{r^2}\frac{\mathrm{d}}{\mathrm{d}\eta};\frac{\partial}{\partial z}=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}\eta};\frac{\partial^2}{\partial r^2}=\frac{z^2}{r^4}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}+\frac{2z}{r^3}\frac{\mathrm{d}}{\mathrm{d}\eta};\frac{\partial^2}{\partial z^2}=\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}\eta^2};\frac{v}{c_0}=k.$$

Then the following equations are obtained

$$\begin{aligned} &(1-\beta)\left[u'(w-\eta u)-u^2-\Phi^2\right]=\eta\pi'+2\pi + k\left[(1+\eta^2)u''+3\eta u'\right]\\ &(1-\beta)\Phi'(w-\eta u)=k\left[(1+\eta^2)\Phi''+3\eta\Phi'\right]\\ &(1-\beta)\left[w'(w-\eta u)-uw\right]=-\pi' + k\left[(1+\eta^2)w''+3\eta w'+w\right];\ w'=\eta u'. \end{aligned} \tag{4.4}$$

It may be easily seen that the boundary conditions become

$$\begin{aligned} &u=\Phi=w=0 && \text{at } \eta=0\\ &u=w=0;\Phi=1,\pi=-\tfrac{1}{2} && \text{at } \eta=\infty. \end{aligned} \tag{4.5}$$

The pressure may be found in a straightforward manner by a single integration of the third equations of system (4.4) over η and satisfying the boundary conditions (4.5)

$$\pi=-(1-\beta)w(w-\eta u)+k\left[(1+\eta^2)w'+\eta w\right]-\tfrac{1}{2}. \tag{4.6}$$

The new variables and function are introduced

$$x=\frac{\eta}{\sqrt{(1+\eta^2)}};y=\frac{1}{\sqrt{(1+\eta^2)}}(w-\eta u) =\sqrt{(1-x^2)}(w-\frac{xu}{\sqrt{(1-x^2)}}. \tag{4.7}$$

Then from the continuity equation we may easily get

$$u=-(1-x^2)y'-xy;\quad w=\sqrt{(1-x^2)}(y-xy') \tag{4.8}$$

where prime denotes differentiation over x. Now system (4.4) is simplified

$$-k(1-x^2)^2y'''=1+(1-\beta)\left[y^2-\Phi^2-\frac{(1-x^2)}{2}(y^2)''-x(y^2)'\right] \tag{4.9}$$

$$k(1-x^2)\Phi''=(1-\beta)y\Phi' \tag{4.10}$$

with the boundary conditions

$$y(0)=0;y(1)=0;y'(0)=0 \tag{4.11}$$

$$\Phi(0)=0;\Phi(1)=1. \tag{4.12}$$

Differentiation of equation (4.9) gives

$$-k(1-x^2)y^{\text{IV}}+4kxy'''=(1-\beta)\times\left[-\frac{2\Phi\Phi'}{(1-x^2)}-\frac{1}{2}(y^2)'''\right].$$

The above result is integrated three times with the original boundary conditions (4.11)

$$-ky'(1-x^2) - 2kxy = -ky''(0)\,x - \tfrac{1}{2}ky'''(0)\,x^2 + (1-\beta)\left[-2\int_0^x\int_0^x\int_0^x \frac{\Phi\Phi'}{(1-x^2)}\,\mathrm{d}x - \frac{y^2}{2}\right]. \tag{4.13}$$

The unknown $y'''(0)$ is to be found from equation (4.9)

$$y'''(0) = -\frac{1}{k}.$$

Then the final result will be

$$2k(1-x^2)\,y' + 4kxy - (1-\beta)\,y^2 = 4(1-\beta)\int_0^x\int_0^x\int_0^x \frac{\Phi\Phi'\,\mathrm{d}x}{(1-x^2)} - x^2 + cx \tag{4.14}$$

with additional notation $2ky''(0) = C$.

The following notation is useful for further manipulations

$$F(x) = 4(1-\beta)\int_0^x\int_0^x\int_0^x \frac{\Phi\Phi'}{(1-x^2)}\,\mathrm{d}x - x^2 + cx. \tag{4.15}$$

The second condition (4.11) implies that $F(1) = 0$.

As regard to the order and sign of $(1-\beta)$, we assume

$$0 < (1-\beta) \leqslant 1. \tag{4.16}$$

Then equation (4.10) may be rewritten as

$$y\Phi' = \frac{k}{(1-\beta)}(1-x^2)\,\Phi''.$$

Integration of the above yields

$$\Phi' = a\exp\int_0^x \frac{(1-\beta)}{k}\frac{y\,\mathrm{d}x}{(1-x^2)};$$

$$\Phi = a\int_0^x\left[\exp\int_0^x \frac{1-\beta}{k}\frac{y\,\mathrm{d}x}{(1-x^2)}\right]\mathrm{d}x. \tag{4.17}$$

The constant a may be found from the last condition (4.12). It is evident that $a = \Phi'(0)$. Further let

$$y = \frac{2k}{(1-\beta)}(1-x^2)\,s(x). \tag{4.18}$$

Then we shall have from (4.14) and (4.17)

$$s' = s^2 + \frac{(1-\beta)}{4k^2}\frac{F(x)}{(1-x^2)^2};\ \Phi = a\int_0^x [\exp\int_0^x 2s\,\mathrm{d}x]\,\mathrm{d}x;$$

$$a = \{\int_0^1 [\exp\int_0^x 2s\,\mathrm{d}x]\,\mathrm{d}x\}^{-1}. \tag{4.19}$$

As far as formula (4.16) is valid, the unequality established by M. A. Goldshtik holds

$$\Phi(x) \leqslant x \text{ at } 0 \leqslant x \leqslant 1. \tag{4.20}$$

Now transform formula (4.15) using the formula for transformation of repeated integral into single integral

$$F(x) = 2(1-\beta)\int_0^x \frac{(x-t)^2}{(1-t^2)}\,\Phi\Phi'\,\mathrm{d}t - x^2 + cx. \tag{4.21}$$

Having found C from the condition $F(1) = 0$, we shall find

$$F(x) = x - x^2 - 2(1-\beta)\left[(1-x)^2 \times \int_0^x \frac{t\Phi^2}{(1-t^2)^2}\,\mathrm{d}t + \int_x^1 \frac{\Phi^2\,\mathrm{d}t}{(1+t)^2}\right]. \tag{4.22}$$

The last two terms in square brackets are strictly positive. Therefore with account for formula (4.16) the following inequality is valid

$$F(x) \geqslant x - x^2 - 2(1-x)^2\int_0^x \frac{t\Phi^2}{(1-t^2)^2}\,\mathrm{d}t - 2x\int_x^1 \frac{\Phi^2\,\mathrm{d}t}{(1+t)^2} \geqslant 0. \tag{4.23}$$

Thus when $0 \leqslant x \leqslant 1$, $F(x) \geqslant 0$; therefore on the basis of Chaplygin's theorem on differential inequalities it follows that $s \geqslant 0$, and consequently, $y \geqslant 0$.

Using the results by M. A. Goldshtik, inequality (4.23) may be specified

$$F(x) \geqslant F_2(x) = \tfrac{1}{2}x(1 - x^2)^2 \text{ at } 0 \leqslant x \leqslant 1. \tag{4.24}$$

Consider the equation

$$\tau' = \tau^2 + \frac{(1-\beta)}{4k^2}\frac{F_2(x)}{(1-x^2)^2}. \tag{4.25}$$

Solution (4.25) that satisfies the condition $\tau(0) = 0$ is of the form

$$\tau = \frac{3}{2}\varkappa\sqrt{(x)}\frac{J_{\frac{2}{3}}(\varkappa x^{\frac{3}{2}})}{J_{-\frac{1}{3}}(\varkappa x^{\frac{3}{2}})};\ \varkappa = \frac{\sqrt{(1-\beta)}}{3k\sqrt{(2)}}. \tag{4.26}$$

From comparison of equation (4.15) with equation (4.19) in view of inequality (4.14) on the basis of Chaplygin's theorem on differential inequalities, it may be concluded that

$$s(x) \geqslant \tau(x) \text{ at } 0 \leqslant x \leqslant 1. \tag{4.27}$$

Function (4.16) is a meromorphic function with the poles at the points

$$x_n = \left(\frac{3k}{(1-\beta)}\sqrt{(2)}\,\mu_n\right)^{\frac{2}{3}} \tag{4.28}$$

where μ_n are the roots of equation $J_{-\frac{1}{3}}(\mu) = 0$. As from the condition, $s(x)$ is a continuous function over the interval (0·1), then for inequality (4.27) to be fulfilled, it is necessary to require that the first pole of function (4.26) would lie outside the interval (0·1), i.e.

$$x_1 > 1 \text{ or } \frac{k}{\sqrt{(1-\beta)}} > \frac{1}{3\sqrt{(2)}\,\mu_1} \approx \frac{1}{8}. \tag{4.29}$$

Inequality (4.29) may be rewritten as

$$Re\sqrt{(1-\beta)} < 8. \tag{4.30}$$

Hence if $\beta = 0$, we arrive at the paradox found originally by M. A. Goldshtik: at the Reynolds numbers above 8, the problem discussed has no finite solution. But because of validity of formula (4.16) at $\beta \neq 0$ for any Re such a parameter may be matched which make inequality (4.30) valid. Thus, the problem discussed is soluable at any Reynolds number within the class of smooth functions. It should be noted that matching of parameter β is equivalent to the choice of the main function which approximates the discontinuity function. As a set of main functions make a space, then the solution obtained should be understood in a generalized sense, i.e. several solutions may be always obtained which will tend in the limit to an ordinary solution. Probably, additional physical considerations are necessary for singling out this extreme solution.

The solution of the boundary-value problem (4.9), (4.10) at the boundary conditions (4.11), (4.12) will be sought using Dorodnitsyn's method of integral relations. Multiplication by the smoothing function $\varkappa(x)$ and integration from zero to unity give the system of integral relations

$$\begin{aligned} -k\int_0^1 \varkappa(x)(1-x^2)^2 y''' \,dx &= \int_0^1 \varkappa(x)\,dx \\ &+ (1-\beta)\int_0^1 \varkappa(x)\left[y^2 - \Phi^2 - \frac{1-x^2}{2}(y^2)''\right. \\ &\left. - x(y^2)'\right]dx;\ k\int_0^1 \varkappa(x)(1-x^2)\Phi''\,dx \\ &= (1-\beta)\int_0^1 \varkappa(x)\,y\,\Phi'\,dx. \end{aligned} \tag{4.31}$$

The first approximation is sought as

$$f_0(x) = 1;\ y = a_1x^2(1-x^2);\ \Phi = x + \beta_1 x(1-x^2). \tag{4.32}$$

Then for determination of unknown coefficients a_1 and b_1, the additional nonlinear system of algebraic equations is used

$$\begin{aligned} 4ka_1 &= 1 + \gamma\left(\frac{8}{105}a_1^2 - \frac{1}{3} - \frac{4}{15}b_1\right) - \frac{8}{105}b_1^2 \\ -\frac{3}{2}kb_1 &= \gamma\left(\frac{2}{15}a_1 - \frac{4}{105}a_1b_1\right) \end{aligned} \tag{4.33}$$

with the additional notation $(1-\beta) = \gamma$.

System (4.33) has been solved by successive approximations using the iteration procedure

$$\left(4k - \frac{8}{105}\gamma a_{1n-1}\right) a_{1n} + \left(\frac{4}{15}\gamma + \frac{8}{105}\gamma b_{1n-1}\right) b_{1n} = \frac{3-\gamma}{3}\left(\frac{2}{15}\gamma - \frac{2}{105}\gamma b_{1n-1}\right) a_{1n} + \left(\frac{3}{2}k - \frac{2}{105}\gamma a_{1n-1}\right) b_{1n} = 0. \quad (4.34)$$

The calculations were carried out on a computer "Promin-2"; these for $Re = 100$ are listed in Table 1.

Any of the values given in Table 1 is a generalized solution of the stated problem differing by the choice of a smoothing function. It should be noted that from the obtained set of generalized solutions, one physical solution must be selected. To do this the obtained solutions should be tested for stability.

Table 1

γ	$0{\cdot}384 \cdot 10^{-2}$	$0{\cdot}448 \cdot 10^{-2}$	$0{\cdot}512 \cdot 10^{-2}$
a_1	32·914	26·112	42·712
b_1	−1·6546	−2·4413	−4·3730

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SUR LES EQUATIONS GENERALISEES DE L'HYDRODYNAMIQUE UTILISEES DANS LA THEORIE DU TRANSFERT THERMIQUE

Résumé—On a présenté dans cet article une nouvelle méthode de composition des solutions générales des équations de l'hydrodynamique. Dans l'établissement de l'équation pour un fluide visqueux on a tenu compte du changement discontinu de la vitesse qui intervient dans les ondes de choc, les écoulements turbulents, les écoulements soniques et supersoniques. Cette discontinuité n'est pas incluse dans les équations du type Navier–Stokes.

La présente méthode est illustrée par la solution du problème sur l'intéraction d'une ligne tourbillon avec la surface. Dans ce cas particulier, les équations d'Euler aboutissent au paradoxe connu de Félix Klein et les équations de Navier–Stokes donnent des solutions paradoxales proposées en premier par M. A. Goldshtik.

La méthode présentée ici élimine les paradoxes mentionnés ci-dessus.

ALLGEMEINE, IN DER WÄRMEÜBERTRAGUNG ANGEWANDTE HYDRODYNAMISCHEN GLEICHUNGEN

Zusammenfassung—Es wird eine neue Methode vorgeschlagen, um allgemeine Lösungen der hydrodynamischen Gleichungen zu erhalten. In der Ableitung der Gleichung eines viskosen Fluides wurde die diskontinuierliche Geschwindigkeitsänderung berücksichtigt; sie kommt in Stosswellen, Wirbelströmungen und Schall- und Überschallströmungen vor. Diese Diskontinuität ist in den hydrodynamischen Gleichungen von Navier–Stokes nicht enthalten. Die vorliegende Methode wird durch die Lösung des Problems der Wechselwirkung zwischen Wirbelfaden und Oberfläche veranschaulicht. In diesem Fall ergeben die Euler-Gleichungen das bekannte Paradoxon von Felix Klein, während die Navier–Stokes-Gleichungen die paradoxen Lösungen ergeben, die von M. A. Goldshtik zuerst untersucht wurden. Die vorliegende Methode eliminiert diese Paradoxa.

ОБ ОБОБЩЕННЫХ РЕШЕНИЯХ УРАВНЕНИИ ГИДРОДИНАМИКИ, ПРИМЕНЯЕМЫХ В ТЕОРИИ ТЕПЛООБМЕНА

Аннотация—В данной работе предлагается новый способ построения обобщенных решений уравнений гидродинамики. Суть способа состоит в том, чтобы при выводе уравнений вязкой жидкости учесть разрывное изменение гидродинамической скорости, что имеет место в ударных волнах, вихревых течениях и при звуковых и сверхзвуковых течениях и что не отражено в уравнениях гидродинамики в форме Навье–Стокса.

Указанный способ продемонстрирован на решении задачи о взаимодействии вихревой нити с поверхностью. Уравнения Эйлера в этом случае приводят к известному парадоксу Феликса Клейна, а уравнения Навье–Стокса дают парадоксальные решения, впервые исследованные М. А. Гольдштиком.

Предлагаемый нами способ построения обобщенных решений устраняет указанные парадоксы.